ALGEBRAIC STRATIFIED GENERAL POSITION AND TRANSVERSALITY

joint work with Clint McCrory and Laurenţiu Păunescu

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Singularity Theory and Regular Stratifications
in honor of David Trotman
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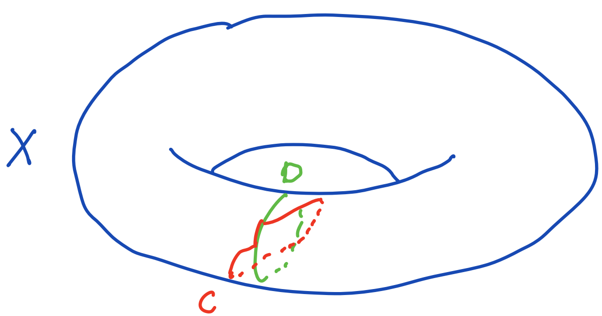
Abstract.

We use the method of Whitney interpolation to construct, for any real or complex projective algebraic variety, a stratified submersive family of self-maps that yields stratified general position and transversality theorems for semialgebraic chains.

This theorem can be used to define an intersection pairing for real intersection homology, an analog of intersection homology for real algebraic varieties.

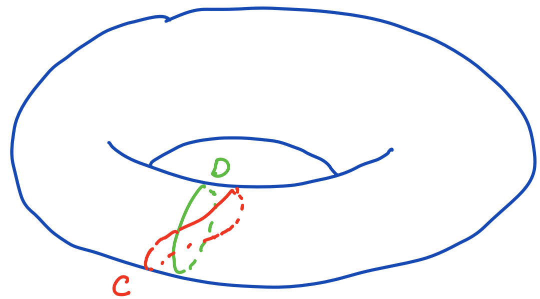
General position. Historical remarks.

- H. Poincaré 1890' and S. Lefschetz 1926 used general position and general transversality to define the intersection product of two cycles on a manifold.



two cyles not in general position

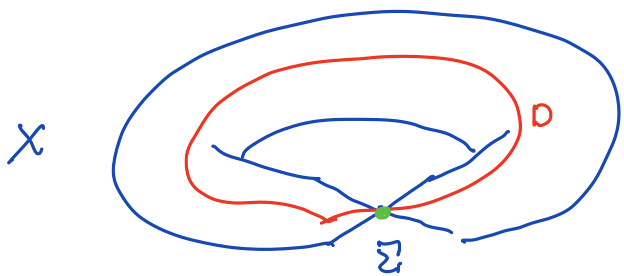
$\dim C \cap D > \dim C + \dim D - \dim X$



two transverse cyles

Stratified general position.

- Stratified general position was used by M. Goresky and R. MacPherson (1980) to define the intersection pairing on Intersection Cohomology of complex algebraic varieties.



D cannot be moved away of Σ even if

$\dim C + \dim \Sigma < \dim X$

McCrory's Theorem (1977).

Let X be a stratified polyhedron and A and B be subpolyhedra.
Then $\exists$ a PL-isotopy of X,

$$\Psi : I \times X \to X,$$

$\Psi_0(x) = \Psi(0, x) = x$, s.t. A and $B' = \Psi_1(B)$ are in general position,
i.e. for every stratum S

$$\dim A \cap B' \cap S \leq \dim A \cap S + \dim B' \cap S - \dim S.$$

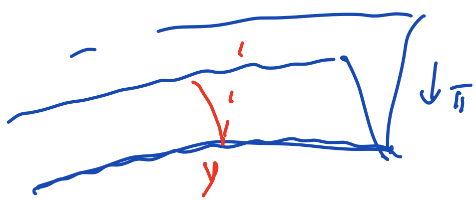
Transversality Lemma of Goresky.

- M. Goresky used startified transversality to construct geometric homology and cohomology theories, for stratified spaces, in which cycles and cocycles are represented by substratified spaces.

Theorem (M. Goresky, 1981)

Let $\mathcal{X}$ be a Whitney stratified subset of a manifold and let $\mathcal{V}$ a Whitney substratified subset satisfying π-fibre condition. Then, each geometric cocycle $\mathcal{Y}$ is cobordant to a geometric cocycle $\mathcal{Y}'$ such that

$$\mathcal{Y}' \text{ is transverse to } \mathcal{V}.$$



Y can be moved

Murolo, du Plessis and Trotman's Theorem.

Theorem (MdPT 2003 & 2005)

Let $\mathcal{X}$ be an abstract stratified set, or a (w) regular stratified subspace of a manifold, and let $\mathcal{V}$ a substratified set. Then, for each substratified set $\mathcal{W}$ of $\mathcal{X}$ there is an isotopy of $\mathcal{X}$ deforming $\mathcal{W}$ to $\mathcal{W}'$ such that

$$\mathcal{W}' \textit{ is transverse to } \mathcal{V}.$$

They give two proofs. One based on Mather's ideas and one on stratified family of diffeomorphisms. (see e.g. Goresky & MacPherson, Stratified Morse Theory)

Submersive family of diffeomorphisms.

Definition

Let P and M be smooth manifolds and let

$$\Psi : P \times M \to M$$

be a smooth mapping. We say Ψ is a submersive family of diffeomorphisms if, for every $(t,x) \in P \times M$, the differential

$$D_t\Psi : T_tP \to T_{\Psi(t,x)}M \text{ is surjective.}$$

i.e. $\Psi^x = \Psi(\cdot, x) : P \times \{x\} \longrightarrow M$ is a submersion for all $x \in M$
and $\Psi_t = \Psi(t, \cdot) : \{t\} \times M \longrightarrow M$ is a diffeo for all $t \in P$.

- Then, by Sard's Theorem, for any submanifolds Z_1, Z_2 of M the set

$$\{t \in P;\ \ \Psi_t(Z_1) \text{ is transverse to } Z_2\ \}$$

is dense in M.

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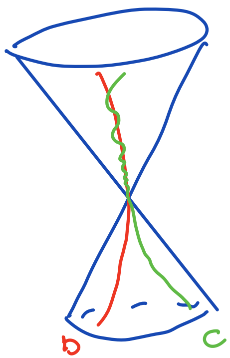
Open problems.

- ("tameness" of intersection near boundary of a stratum)
  Even if $\mathcal{W}' \cap S$ and $\mathcal{V} \cap S$ are of complementary dimension and transverse in S it is not clear that $\mathcal{W}' \cap \mathcal{V} \cap S$ if finite.



2 cycles transverse but $C \cap D$ is infinite.

- (preservation of regularity after deformation)
  If the given stratification of $\mathcal{W}$ is (Whitney, (w), (a), Bekka, etc.) regular.
  Does the stratification of $\mathcal{W}'$ inherit the same regularity?

Stratified submersive family of diffeomorphisms.

For a stratified set $X = \bigsqcup S_i$ we say that

$$\Psi : P \times X \to X$$

is a stratified submersive family of diffeomorphisms if for every stratum S_j, $\Psi(P \times S_j) \subset S_j$ and the restriction $\Psi_{|S_j} : P \times S_j \to S_j$ is a submersive family of diffeomorphisms.

Algebraic stratified general position.

Theorem (C. McCrory, A.P., L. Păunescu, J. Alg. Geometry, 2019)

Let $\mathcal{V} = \{V_i\}$ be a finite family of algebraic subsets of $\mathbb{P}^n(\mathbb{K})$. There exists an algebraic stratification $\mathcal{S} = \{S_j\}$ of $\mathbb{P}^n$ compatible with each V_i and a semialgebraic stratified arc-wise analytic submersive family of diffeomorphisms

$$\Psi : U \times \mathbb{P}^n \to \mathbb{P}^n,$$

where U is a neighborhood of the origin in $\mathbb{K}^{n+1}$, and $\Psi(0, x) = x$ for all $x \in \mathbb{P}^n$.

In particular, $\Phi(t, x) = (t, \Psi(t, x)) : U \times \mathbb{P}^n \to U \times \mathbb{P}^n$ is an arc-wise analytic trivialization

- Φ is semi-algebraic and stratified real analytic isomorphism.
- Φ and Φ^{-1} are real analytic on real analytic arcs.
- Φ is $\mathbb{K}$-analytic in t.

Corollary

Let Z and W be semialgebraic subsets of $\mathbb{P}^n$. There is an open dense semialgebraic subset U' of U such that, for all $t \in U'$ and all strata $S \in \mathcal{S}$,

$$\dim(Z \cap \Psi_t(W) \cap S) \leq \dim(Z \cap S) + \dim(W \cap S) - \dim S.$$

If $(Z, \mathcal{A})$ and $(W, \mathcal{B})$ are stratified semialgebraic subsets of $(\mathbb{P}^n, \mathcal{S})$ then there is an open dense semialgebraic subset U' of U such that, for all $t \in U'$, and all strata $S \in \mathcal{S}$,

$$(Z \cap S, \mathcal{A}) \pitchfork (\Psi_t(W) \cap S, \Psi_t(\mathcal{B})) \qquad \textit{in } S.$$

Arc-wise analytic equisingularity.

Simplifying assumptions:

- $F(t, x)$ homogeneous polynomial in $x \in \mathbb{K}^{n+1}$, analytic in parameter $t \in U$;
- system of coordinates x (linearly) sufficiently generic;
- the family of zero sets of $F(t, x)$, $t \to V_t$ Zariski equisingular.

Recipe for constructing arcwise analytic homeomorphism trivializing $t \to V_t$

$$\Phi(t, x) : (U, 0) \times \mathbb{K}^{n+1} \to (U, 0) \times \mathbb{K}^{n+1}.$$

- Let $F_{n+1}(t, x) := F(t, x)$,
  Define recursively $F_j(t, x_1, \ldots, x_j) = \Delta_{x_{j+1}} (F_{j+1})_{red}$.
- Then construct
  $$\Phi_j(t, x_1, \ldots, x_j) : U \times \mathbb{K}^j \to U \times \mathbb{K}^j$$
  by induction on j.

Some details.

Let

$$F_{n+1}(t,x) = \prod_i (x_{n+1} - \xi_i(t,x'))$$

$x' = (x_1, \ldots, x_n)$. ξ_i: the roots of F_{n+1}.

Given $\Phi_n(t,x') = (t, \Psi_n(t,x'))$.
Construct $\Phi_{n+1}(t,x) = (t, \Psi_{n+1}(t,x)) = (t, \Psi_n(t,x'), \psi_{n+1}(t,x))$.

1. **Lift** $\Phi_n(t,x')$ to the zero set of $F_{n+1}(t,x)$

$$\psi_{n+1}(t,x',\xi_i(0,x')) = \xi_i(\Phi_n(t,x')).$$

2. **Extend** to arbitrary (t, x', x_{n+1})

$$\psi_{n+1}(t,x',x_n) = \psi(\xi(0,x'), \xi(\Phi_n(t,x')), x_n),$$

where $\xi = (\xi_1, \ldots, \xi_{\deg F_{n+1}})$, ψ is a Whitney Interpolation function. In particular ψ is real rational function given by a precise formula.

Main idea.

Let $F(x)$ be a homogeneous polynomial (not depending) on $t \in U$. Then the family of its zero sets $t \to V_t$ is trivial and the identity map

$$id : U \times \mathbb{K}^{n+1} \to U \times \mathbb{K}^{n+1}$$

trivializes it.

Construct, recursively on j, a non-trivial trivialization of this trivial family

$$\Phi_j(t_1, \dots, t_j, x_1, \dots, x_j) : U_j \times \mathbb{K}^j \to U_j \times \mathbb{K}^j,$$

by introducing, at each step, to the precise formula for Φ_j a new parameter t_j.

Perturbed formula for Whitney's Interpolation:

$$\psi(a,b,z,\eta) = z - \frac{\sum_i \mu_i(a,z)(b_i - a_i)|z|^d + \eta z}{\sum_i \mu_i(a,z)|z|^{-d} + 1}$$

new terms

where η is the perturbation parameter

d homogenity degree of each μ_i

$$\frac{\partial \psi}{\partial \eta} = \frac{z|z|^d}{\sum \mu_i + |z|^d} = 0 \quad \text{iff } z = 0 \text{ or } z = a_i.$$

Many happy returns, David!