The intrinsic stratification of a semialgebraic set

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Prelude

As a young PL topologist, I was puzzled by Hironaka's statement that there is a "canonical PL structure" on a semialgebraic set (*Triangulations of algebraic sets* (1975), pages 165, 178).

His reason was that two semialgebraic triangulations have a common semialgebraic subdivision.

But if you could conclude from this that the triangulations are piecewise-linearly isomorphic, you would have a very short proof of the semialgebraic Hauptvermutung!

Semialgebraic triangulation

Let $X \subset \mathbb{R}^N$ be a compact semialgebraic set. A *semialgebraic triangulation* of X is a finite Euclidean simplicial complex K together with a semialgebraic homeomorphism $f : |K| \to X$.

A (compact) *polyhedron* P is the union of the simplices of a finite euclidean simplicial complex: $P = |K|$. (Thus P is a semialgebraic set.)

A *piecewise-linear (PL) homeomorphism* of polyhedra is a homeomorphism $\varphi : P \to Q$ such that the graph of φ is a polyhedron.

Semialgebraic Hauptvermutung

Theorem (Shiota-Yokoi 1984)

If two polyhedra are semialgebraically homeomorphic, then they are piecewise-linearly homeomorphic.

Thus if $f : |K| \to X$ and $g : |L| \to X$ are semialgebraic triangulations of the compact semialgebraic set X, there is a PL homeomorphism $\varphi : |K| \to |L|$. (In other words, K and L have isomorphic simplicial subdivisions.)

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I'll show how to deduce the theorem from the following special case. This argument is one step in Shiota and Yokoi's proof.

Theorem (Hauptvermutung for a simplex)

If a polyhedron is semialgebraically homeomorphic to a Euclidean simplex Δ, then it is PL homeomorphic to Δ.

Proof of the Hauptvermutung (1/2)

Let $h : P \to Q$ be a semialgebraic homeomorphism of polyhedra, with $P = |K|$ and $Q = |L|$.

First we use Hironaka's idea! There is a semialgebraic triangulation $g : |M| \to Q$ that is a common semialgebraic subdivision of the triangulations $h : |K| \to Q$ and $|L| = Q$. Let $f = h^{-1} \circ g$. We have for all simplices $\Gamma \in |K|$ and $\Delta \in |L|$, the semialgebraic sets $f^{-1}(\Gamma)$ and $g^{-1}(\Delta)$ are subpolyhedra of $R = |M|$.

If we show that R is PL homeomorphic to P and Q, then P is PL homeomorphic to Q. We will use the semialgebraic homeomorphism $g : R \to Q$ to prove that R is PL homeomorphic to Q. The same argument applies to $f : R \to P$.

Proof of the Hauptvermutung (2/2)

Now $g : R \to Q$ is a semialgebraic homeomorphism of polyhedra, with $Q = |L|$, such that for all simplices $\Delta \in L$, the semialgebraic set $g^{-1}(\Delta)$ is a subpolyhedron of R.

By the Hauptvermutung for a simplex, for every simplex $\Delta \in L$, the polyhedron $g^{-1}(\Delta)$ is PL isomorphic to Δ. In other words, if Δ is a k-simplex, then $B_\Delta = g^{-1}(\Delta)$ is a PL k-ball.

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Let $Q_k = |L_k|$, where L_k is the k-skeleton of L, and let $R_k = g^{-1}(Q_k)$, the k-skeleton of the PL cell complex $\{B_\Delta \mid \Delta \in L\}$. We construct a PL homeomorphism $\varphi : R \to Q$ by induction on k: $\varphi_k = \varphi|R_k : R_k \to Q_k$.

Let $\varphi_0 = g|R_0$. If φ_{k-1} is defined, we extend it to φ_k using that for each k-simplex Δ, the PL k-ball B_Δ is the cone on its boundary. We define $\varphi_k|B_\Delta$ to be the cone on $\varphi_{k-1}|\partial B_\Delta$. □

Refined Hauptvermutung

This method of proof gives more:

Theorem (Shiota-Yokoi)

Let $h : P \to Q$ be a semialgebraic homeomorphism of polyhedra. Given $\epsilon > 0$, there is a semialgebraic isotopy $h_t : P \to Q$, $0 \leq t \leq 1$, such that $h_0 = h$, $h_1 = \varphi$ is a PL homeomorphism, and $|h_t(p) - h(p)| < \epsilon$ for all (p, t).

Moreover, if $h|S$ is PL for a subpolyhedron $S \subset P$, then the isotopy h_t can be chosen so that $h_t(p) = h(p)$ for all $p \in S$ and all t.

Proof of the refined Hauptvermutung (1/3)

Lemma (Alexander Trick)

Let $f : B \to B$ be a homeomorphism of a ball B, and let f_t' be an isotopy of ∂B such that $f_0' = f|\partial B$. There is an isotopy f_t of B such that $f_0 = f$, $f_t|\partial B = f_t'$, and f_1 is the cone on f_1'.

Proof: The ball $B \times I$ is the cone on $(B \times \{0\}) \cup (\partial B \times I)$, and a suitable isotopy $f_t : B \to B$ is the cone on the self-homeomorphism of $(B \times \{0\}) \cup (\partial B \times I)$ given by $(x, 0) \mapsto (f(x), 0)$ and $(x, t) \mapsto (f_t'(x), t)$. $\square$

Proof of the refined Hauptvermutung (1/3)

Lemma (Alexander Trick)

Let $f : B \to B$ be a homeomorphism of a ball B, and let f'_t be an isotopy of ∂B such that $f'_0 = f|\partial B$. There is an isotopy f_t of B such that $f_0 = f$, $f_t|\partial B = f'_t$, and f_1 is the cone on f'_1.

Proof: The ball $B \times I$ is the cone on $(B \times \{0\}) \cup (\partial B \times I)$, and a suitable isotopy $f_t : B \to B$ is the cone on the self-homeomorphism of $(B \times \{0\}) \cup (\partial B \times I)$ given by $(x, 0) \mapsto (f(x), 0)$ and $(x, t) \mapsto (f'_t(x), t)$. □

This is true in the topological, semialgebraic, or piecewise-linear categories – but not in the differential category. It implies, for example, that if $f : B \to B$ is a homeomorphism of a ball such that $f|\partial B$ is the identity, then f is isotopic rel ∂B to the identity.

Proof of the refined Hauptvermutung (2/3)

We apply the Alexander trick to the semialgebraic homeomorphism $g : R \to Q$ in the proof of the Hauptvermutung. We construct an isotopy g_t from g to φ by induction. For each k we define an isotopy $(g_k)_t : R_k \to Q_k$ such that $(g_k)_0 = g|R_k$ and $(g_k)_1 = \varphi_k$, the restriction to R_k of the PL homeomorphism $\varphi : R \to Q$.

Recall that $Q = |L|$, and for every simplex $\Delta \in L$, g restricts to a semialgebraic homeomorphism $g_\Delta : B_\Delta \to \Delta$. Suppose that $(g_{k-1})_t : R_{k-1} \to Q_{k-1}$ has been defined. For each k-simplex Δ, we apply the Alexander trick to g_Δ and the isotopy $(g_{k-1})_t|\partial B_\Delta$ to obtain $(g_k)_t|B_\Delta$. Since $(g_{k-1})_1 = \varphi_{k-1}$, we have that $(g_k)_1$ is the cone on $\varphi_{k-1}|\partial B_\Delta$, which is $\varphi_k|B_\Delta$.

Proof of the refined Hauptvermutung (3/3)

The desired properties of the isotopy g_t are straightforward consequences of the construction:

The isotopy $g_t : R \to Q$ can be made arbitrarily small simply by starting with a fine triangulation $|L| = Q$, since for every simplex $\Delta \in L$ and all t, we have $g_t(B_\Delta) = \Delta$.

Furthermore, if the restriction of $h : P \to Q$ to a subpolyhedron is PL, we can assume that there is a subcomplex N of M such that for each simplex $\Sigma \in N$ there is a simplex $\Delta \in L$ such that $g : |M| \to Q$ restricts to a linear isomorphism $g_\Delta : \Sigma \to \Delta$. Then we can inductively define the PL homeomorphism φ and the isotopy g_t so that, for all such pairs (Σ, Δ), we have $\varphi|\Sigma = g_\Delta = g_t|\Sigma$ for all t. □

Local Semialgebraic Hauptvermutung

Corollary

If $h : P \to Q$ *is a semialgebraic homeomorphism of polyhedra, then for all* $p \in P$*, the link of* p *in* P *is PL homeomorphic to the link of* $h(p)$ *in* Q*:*

$$Link(P, p) \cong Link(Q, h(p)).$$

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$$Link(P, p) \cong Link(Q, h(p)).$$

Thus if $f : P \to X$ is a semialgebraic homeomorphism and $x \in X$, the polyhedron $Link(P, f^{-1}(x))$ is independent of P and f, up to PL homeomorphism. Moreover, this PL link is semialgebraically homeomorphic to $L(X, x)$, the *semialgebraic link* of x in X, which is defined as the intersection of $X \subset \mathbb{R}^N$ with a sufficiently small ϵ-sphere centered at x, $S^{N-1}(x, \epsilon) \subset \mathbb{R}^N$. It follows that $L(X, x)$ is a semialgebraic invariant of X, a result of Coste and Kurdyka (1992).

Intrinsic stratification of a polyhedron

A polyhedron P has a canonical filtration by subpolyhedra

$$P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n = P,$$

where P_k is the intersection of the k-skeletons of all PL triangulations (Zeeman). The set $P_k \setminus P_{k-1}$ is a PL k-manifold, and the resulting decomposition of P is the *intrinsic stratification* of P. If $p \in P_k \setminus P_{k-1}$, we say P has *intrinsic dimension* k at p: $d(P, p) = k$.

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Theorem (Armstrong 1967)

Let P be a polyhedron, and $p \in P$. Then $d(P, p) \geq k$ if and only if $Link(P, p)$ is a k-fold suspension, i.e. $Link(P, p)$ is PL homeomorphic to $S^k(A)$ for some polyhedron A.

NOTE: In the PL category, $S(A) \cong S(B)$ implies $A \cong B$. *This is false in the topological category.*

Intrinsic stratification of a semialgebraic set

We define the *intrinsic filtration* of a semialgebraic set X,

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X,$$

by $X_k = f(P_k)$, where P is a polyhedron and $f : P \to X$ is a semialgebraic homeomorphism.

This filtration is well-defined by Armstrong's theorem and the local semialgebraic Hauptvermutung.

It follows from the definition that X_k is the intersection of the k-skeltons of all semialgbraic triangulations of X.

Properties of the intrinsic stratification

Let X be a semialgebraic set with intrinsic filtration

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X.$$

The intrinsic filtration defines a *semialgebraic stratification* of X:

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- For each k the set $X_k \setminus X_{k-1}$ is a semialgebraic k-manifold; *i.e.* it is locally semialgebraically homeomorphic to an open semialgebraic subset of $\mathbb{R}^k$.
- The connected components of the sets $X_k \setminus X_{k-1}$ are the *strata* of X. The strata satisfy the *frontier condition*: If Y and Z are strata with $Y \cap \overline{Z} \neq \emptyset$, then $Y \subset \overline{Z}$.

Properties of the intrinsic stratification

The intrinsic stratification of X is characterized by the fact that two points x, y lie in the same stratum if and only if there is a semialgebraic isotopy h_t of X such that h_0 is the identity and $h_1(x) = y$.

If such an isotopy exists, then x and y are in the same intrinsic stratum, because the intrinsic filtration is invariant under semialgebraic homeomorphism.

Given two points in the same stratum, the existence of an isotopy connecting them follows from the fact that the intrinsic stratification is *locally semialgebraically trivial*.

Local semialgebraic triviality

Let X be a filtered semialgebraic set

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that defines a semialgebraic stratification of X. We say that the resulting stratification is *locally (semialgebraically) trivial* if, for every $x \in X_k \setminus X_{k-1}$ there exists a filtered semialgebraic set

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{n-k-1} = A,$$

a closed semialgebraic neighborhood U of x in X, and a semialgebraic homeomorphism $h : U \to \mathbb{B}^k \times c(A)$, $\mathbb{B}^k$ the unit ball in $\mathbb{R}^k$, such that $h(U \cap X_k) = \mathbb{B}^k \times c$, and for all $\ell > k$, $h(U \cap X_\ell) = \mathbb{B}^k \times c(A_{\ell-k-1})$.

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(Compare Siebenmann's *CS sets*, or *locally cone-like topological stratified sets*.)

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For a locally trivial semialgebraic stratification of X, there is a *stratified general position theorem* for pairs of closed semialgebraic subsets of X. This is the semialgebraic version of the PL stratified general position theorem (McCrory 1972).

Nash stratifications and local triviality

A *Nash stratification* of a semialgebraic set $X \subset \mathbb{R}^N$ is a semialgebraic stratification such that the strata are Nash submanifolds of $\mathbb{R}^N$. (Every point in a k-stratum Y has a semialgebraic open neighborhood in $(\mathbb{R}^N, Y)$ that is Nash diffeomorphic to an open semialgebraic subset of $(\mathbb{R}^N, \mathbb{R}^k)$.)

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Coste and Shiota (1995) proved that every semialgebraic Whitney stratification (a Nash stratification satisfying Whitney's condition (b)) is locally semialgebraically trivial. The converse is false: A locally semialgebraically trivial Nash stratification is not necessarily Whitney. There is a real algebraic example in David's 1983 Arcata survey article on regularity conditions for pairs of strata (pages 579-80). His example is locally semialgebraically trivial, but Whitney's condition (a) fails.

Complex varieties and local triviality

It is not known whether every locally topologically trivial complex algebraic stratification satifies Whitney's condition (a), but Briançon and Speder (1975) produced a complex algebraic example that is locally topologically trivial but does not satisfy Whitney's condition (b).

David suggested to me that the Briançon-Speder example should also be locally semialgebraically trivial. That turns out to be true. David pointed out that, since their example is quasi-homogeneous, it satisfies Bekka's condition (c). Coste and Shiota assert that their proof is valid in that case: A semialgebraic Bekka (c) stratification is locally semialgebraically trivial.

Addendum: Ambient local triviality

Note that the cuspidal cubic $\{y^2 = x^3\} \subset \mathbb{C}^2$ has only one intrinsic stratum, even though it is locally knotted at the origin.

To compare locally semialgebraically trivial stratifications with Nash stratifications of $X \subset \mathbb{R}^N$, it is natural to include the stratum $\mathbb{R}^N \setminus X$.

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If one considers semialgebraic triangulations of $(\mathbb{R}^N, X)$, one obtains an "ambient" intrinsic stratification of X, *i.e.* an intrinsic stratification of the pair $(\mathbb{R}^N, X)$. The strata are locally unknotted (*locally flat*) in $\mathbb{R}^N$, and the stratification satisfies an ambient version of local semialgebraic triviality, which includes the stratum $\mathbb{R}^N \setminus X$.

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The Coste-Shiota theorem and the preceding examples are valid for this stronger version of local semialgebraic triviality.

Question about real algebraic sets

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Question about real algebraic sets

Can Akbulut and King's *Topology of Real Algebraic Sets* (1992) be recast in the semialgebraic category?

(1) Every compact PL manifold X is homeomorphic to a real algebraic set. Is X *semialgebraically* homeomorphic to a real algebraic set?

(2) Let X be a compact 3-dimensional semialgebraic set such that the *Akbulut-King numbers* vanish for all the links in X. Then X is homeomorphic to an algebraic set Y. The topological Hauptvermutung for 3-complexes (E. M. Brown 1969) implies that semialgebraic triangulations of X and Y are PL homeomorphic, so X and Y are semialgebraically homeomorphic.

Thus every 3-dimensional semialgebraic set with vanishing Akbulut-King numbers is *semialgebraically* homeomorphic to a real algebraic set. Can this result be proved without the topological Hauptvermutung?

THANK YOU

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CONGRATULATIONS TO DAVID !