Holomorphic mappings and Milnor fibrations

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In singularity theory, there are certain well-known facts about holomorphic functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. What happens if we replace $(\mathbb{C}, 0)$ by $(\mathbb{C}^k, 0)$, $k \leq n$?
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c) $R^l(f|\{\|z\| \leq \epsilon, |f| < \alpha\})_*\mathbb{Z}$ is locally constant on $\{0 < |t| < \alpha\}$.
This enables the introduction of the (cohomological) monodromy, as it follows from b).

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b) not at all evident.
Need: S_ϵ intersects all $\{f = t\}$, $0 < |t| \ll \epsilon \ll 1$, transversally.
But there is a stratification of $(\mathbb{C}^n, 0)$ which satisfies Thom's a_f condition, by Hironaka.
In fact, Whitney's condition b implies a_f in the complex case, by Briançon-Maisonobe-Merle.

3. Now replace $\mathbb{C}$ by $\mathbb{C}^k$, $n \geq k > 1$. First: case of an isolated singularity.
What does this mean? Impossible: f has an isolated critical point.
So suppose instead: $f^{-1}(0)$ is an ICIS: i.e. has dimension $n - k$ and has an isolated singular point.
Then: $f : C \to \{\|t\| < \alpha\}$ is finite, where $C :=$ set of critical points of f in $\{\|z\| < \epsilon, \|f(z)\| < \alpha\}$.
So $f(C)$ analytic subset of $(\mathbb{C}^k, 0)$, in fact, a hypersurface.

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In particular, we have an action of the fundamental group $\pi_1(\{\|t\| < \alpha\} \setminus f(C))$ on the cohomology groups of the general fibre (monodromy).

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Take representative of C. If $0 < \epsilon' < \epsilon \ll 1$: $C \cap B_{\epsilon'}, C \cap B_\epsilon$ define same germ, but what about $f(C \cap B_{\epsilon'}), f(C \cap B_\epsilon)$? Here o.k.: $f(C \cap B_{\epsilon'}) \cap \{|t\| < \alpha\} = f(C \cap B_\epsilon) \cap \{|t\| < \alpha\}$ if $0 < \alpha \ll \epsilon' < \epsilon \ll 1$.

But for other analytic subgerms there might be a problem.

4. Finally pass to the general case $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$, $n \geq k$. Then the geometry of the mapping can be very complicated. As it was known to R. Thom, this happens in particular with (non-trivial) blowing-up,
that is why he introduced the notion of "morphisme sans éclatement". The Thom a_f condition fails, whatever stratification we take.
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Easiest example of a blowing-up mapping:
$f : \mathbb{C}^2 \to \mathbb{C}^2 : (z_1, z_2) \mapsto (z_1, z_1 z_2)$.
a) must fail: $f(\mathbb{C}^2) = (\mathbb{C}^* \times \mathbb{C}) \cup \{0\}$.

Now look at $f(B_\epsilon)$: it is compact semialgebraic. More precisely:
Image of $\{|z_1|^2 + |z_2|^2 \leq \epsilon^2\}$: $\{|w_1|^2 + |w_2/w_1|^2 \leq \epsilon^2\} \cup \{0\}$,
because $w_1 = z_1, w_2 = z_1 z_2$,
i.e. $|w_1|^4 + |w_2|^2 \leq \epsilon^2 |w_1|^2$, or: $|w_2| \leq |w_1|\sqrt{\epsilon^2 - |w_1|^2}$.
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The germ of $f(B_\epsilon)$ at 0 moves with ϵ, and it is not complex analytic.
In fact, this "moving" is responsible for the image not being complex analytic, as we will see.

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But it is more difficult to tell something in the case of a non-open mapping. We will prove:

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Suppose that the image of $(\mathbb{C}^n, 0)$ is a well-defined germ. Then this germ is complex analytic.

This generalizes [JT1] Prop. 2.1 from $k = 2$ to arbitrary n. Note that the germ must then be irreducible, of course.

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a) For all $x \in A$ and all irreducible complex curve germs Γ in U at x: $\Gamma \cap A = \{x\}$ or $\Gamma \subset A$,
b) A is complex analytic.

Recall our claim:

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Proof: b) $\Rightarrow$ a): clear.

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Proof: b) $\Rightarrow$ a): clear.
a) $\Rightarrow$ b): Let $x \in A_{reg}$, $v \in T_x A_{reg}$, represented by a real analytic curve germ in A. Let Γ be its complexification. Then $\Gamma \cap A \neq \{x\}$, hence $\Gamma \subset A$. So $J(v) \in T_x A_{reg}$, too, hence A_{reg} is an almost complex submanifold of U. By the Newlander-Nirenberg theorem, the integrability condition is fulfilled, so A_{reg} is a complex submanifold of U.

We cannot conclude directly that A must be complex analytic: Take a closed disc in the complex plane, then the regular part is the open disc, hence a submanifold of $\mathbb{C}$, but the closed disc is not complex analytic. So we must use our hypothesis again.

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We choose a subanalytic Whitney stratification of A. We know already that dim A is even, so let dim $A = 2m$. Assume that there is a stratum S of dimension $2m-1$. Let $x \in S$. The $v \in T_xS$ such that $J(v) \in T_xS$ form a vector space of complex dimension s. Choose s general complex linear functions in order to reduce to the case $s = 0$.

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Then there is an embedding $\gamma : I^{2m-1} \to S$ where I is an open symmetric interval. Let D be the corresponding disc, then by complexification we get an embedding $\Gamma : D^{2m-1} \to \mathbb{C}^n$. Our hypothesis implies by induction that $\Gamma(D^d \times I^{2m-d-1})$, $d = 0, \ldots, 2m-1$, is contained in A, so $\Gamma(D^{2m-1}) \subset A$, hence $4m - 2 \leq 2m$, i.e. $m = 1$.

So dim $S = 1$, dim $A = 2$, $\Gamma(D)$ is a smooth complex curve in A, hence $\Gamma(D)$ consists of two local branches of A (which is considered to be branched along S). Suppose there is still another one: then we find another irreducible complex curve which intersects $\Gamma(D)$ along S which is impossible.

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Now return to our original situation. Let A' be the closure of the union of all strata of dimension $2m$. Then we get that the strata of dimension $2m-1$ are superfluous for A', so A'_{sing} has real codimension ≥ 2 in A.

Since A'_{reg} is complex analytic of pure dimension m we obtain that $\overline{A'}$ is complex analytic, by the

Theorem of Remmert-Stein-Shiffman: Suppose that X is a complex space, A a closed complex analytic subset of $X \setminus B$ of pure dimension m, where B is closed in X and the $2m-1$-dimensional Hausdorff measure of B is 0, then $\bar{A}$ is a purely m-dimensional complex analytic subset of X.

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Note that the condition on B is fulfilled, in particular, if B is complex analytic of dimension $< m$: Theorem of Remmert-Stein.

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Otherwise argue similarly as before, by reduction to the case $k = 1$, so that S is superfluous.

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Continuing we conclude that A is the union of complex analytic sets, hence complex analytic itself.

Recall the result announced before, in a generalized form:

Suppose that $(B, 0)$ *is a complex analytic subgerm of* $(\mathbb{C}^n, 0)$ *and that the image of* $(B, 0)$ *is a well-defined germ. Then the image germ is complex analytic.*

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Proof: Let $0 < \alpha \ll \epsilon \ll 1$ and $x \in \mathbb{C}^k$, $\|x\| < \alpha$. Let Γ be an irreducible complex curve germ at x. Look at $f|f^{-1}(\Gamma) \cap B \cap B_\epsilon$. We may suppose now that S_ϵ intersects $f^{-1}(x) \cap B$ transversally in the stratified sense with respect to some Whitney stratification of $f^{-1}(\Gamma) \cap B$ which is compatible with $f^{-1}(x) \cap B$. Then S_ϵ intersects all nearby fibres of $f|B$ transversally, too, so $f(f^{-1}(\Gamma) \cap B \cap B_\epsilon) = \Gamma \cap f(B \cap B_\epsilon)$ is open in Γ or $= \{x\}$. Now apply the valuative criterion.

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In this case: the image germ of $(\mathbb{C}^n, 0)$ is $(\mathbb{C}^k, 0)$ resp. $(D, 0)$ if the fibres are non-empty resp. empty.

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Condition b) is similar to the tameness condition in [JT2] (but not the same).

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The example of the blowing-up mapping above shows that it is not sufficient to assume that the germ of $f(C)$ is well-defined: there, $C = \{0\} \times \mathbb{C}$, so $f(C) = \{0\}$.