# On the $t^r$ condition

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Singularity Theory and Regular Stratifications,
Celebration of David Trotman's Retirement,
Marseille, October 1, 2021

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In general,(b) and (w) are independent; for $\mathbb{C}$ analytic sets, they are equivalent; for real analytic sets, (w) implies (b), but the converse fails (example $y^4 = t^4x + x^3$ by Brodersen-Trotman).

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Condition (w) is a strengthening of Whitney's condition ($a$): $d(T_xX, T_yY) \to 0$ as $x \to y$.

Condition ($a$) does not imply topological equisingularity, as we see in the example:

## Whitney's Example satisfying (a) but not topologically equisingular

$$x^2 + z^2(z - y^2) = 0$$

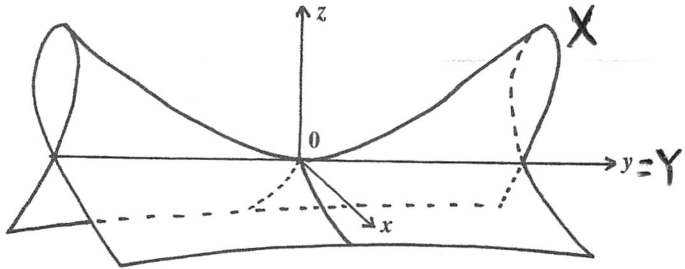


FIG. 1.

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**Theorem Kuo 1978**

If $(X, Y)$ is $(a)$-regular at $y \in Y$ then $(h^\infty)$ holds, i.e. the germs at $y$ of intersections $S \cap X$, where $S$ is a $C^\infty$ submfd transverse to $Y$ at $y$ and $\dim S + \dim Y = \dim M$ ($S$ is called a *direct transversal*) are homeomorphic.

## Definition: Trotman's refinement of Thom's condition

$(X, Y)$ is $(t^r)$-regular $(r \geq 1)$ at $y \in Y$ if every $C^r$ submfd $S$ transverse to $Y$ at $y$ is transverse to $X$ nearby.

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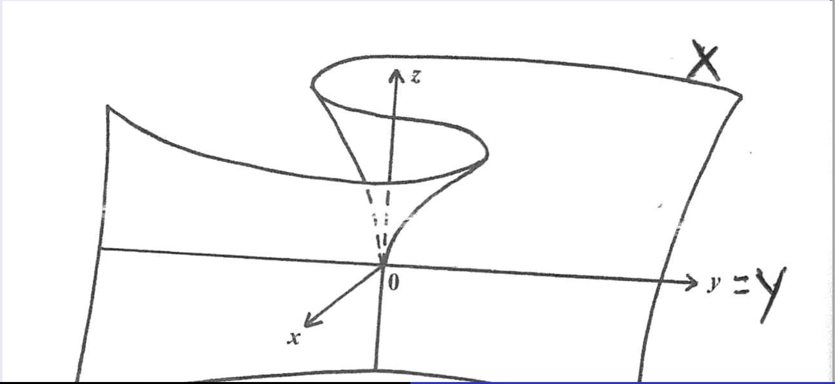
**Theorem (Trotman 1985)**

$(t^1)$ is equivalent to $(h^1)$.

## Example of Koike-Kucharz

$x^3 - 3xz^5 + yz^6 = 0$, with $Y$ the $y$-axis and $X = Z - Y$. $(X, Y)$ is $(t^2)$, not $(t^1)$. There are two topological types of germs at 0 of intersections $S \cap X$ where $S$ is a $C^2$ submfd transverse to $Y$ at 0. However the number of topological types of such germs for $S$ of class $C^1$ is uncountable.



## Trotman-Wilson 1999

For subanalytic strata ($t^r$) is equivalent to the finiteness of the number of topological types of germs at $y$ of $S \cap X$ for $S$ a $C^r$ transversal to $Y$ ($1 \leq r \leq \infty$).

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## Kuo-Trotman 1988, Trotman-Wilson 1999

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A definition of $(t^r)$ is given in TW so that $(t^0)$ is equivalent to $(w)$. So $(t^1) \implies (h^1)$ follows from Grassman blowup and then applying the Verdier Isotopy Theorem.

## Generalizations of $(t^r)$

Assume $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^p$, $Y = 0 \times \mathbb{R}^p$, $X \subset (\mathbb{R}^m - Y)$, $Y \subset \overline{X}$, $X \cup Y$ locally closed. Any direct transversal to $Y$ at 0 is the graph of a function germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p$.

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Such an $f$ or its transversal $\Gamma f$ satisfies $(t^{r,s})$ for $X$ if there does not exist any sequence of 1-jets $(x_i, y_i, L_i)$ in $J^1(\mathbb{R}^n, \mathbb{R}^p)$ such that $y_i = o(\|x_i\|^r)$ and $L_i = o(\|x_i\|^s)$, $p_i = (x_i, f(x_i) + y_i) \in X$, and $df(x_i) + L_i$ is non-transverse to $T_{p_i}X$. Alternatively this can be stated as $f$ satisfying a Łojasiewicz inequality for how close it is from being non-transverse to $X$.

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$X$ is said to satisfy $(t^{r,s})$ if all $f$ (or $\Gamma f$) satisfy $(t^{r,s})$ for $X$, $f$ a polynomial map of degree at most $[r]$ = the greatest integer less or equal to $r$. Condition $(t^r)$ as defined before is equivalent to $(t^{r,r-1})$, $(w)$ is equivalent to $(t^{0,-1})$, and $(a)$ is equivalent to $(t^{0,0-})$ (where the minus means replace “little oh" by “big oh" in the definition above). The Koike-Kucharz example satisfies $(t^r)$ iff $r > 1.5$.

## More generalizations of $(t^r)$

The $k$-jet $z$ of a $C^r$ transversal $\Gamma f$ ($k \leq r$) is the set of graphs of functions in $j^k f$. Then "$z$ satisfies $(t^r)$" if all the transversals in $z$ are transverse to $X$, equivalently if any representative of $z$ satisfies $(t^{r,r-1})$ for $X$.

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In the definition of $(t^r)$ or $(t^{r,s})$ the single stratum $X$ can be replaced by a whole stratification, or more generally, by any finite or infinite collection $\{X_a\}$ of submanifolds which partition a locally closed subset $K$ of the complement of $Y$.

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The $k$-jet $z$ is "$\mathcal{S}$-sufficient" (resp. "weakly $\mathcal{S}$-sufficient") if for all $S, T \in z$ there is a homeomorphism germ from $S$ to $T$ (resp. $S \cap K$ to $T \cap K$) preserving all intersections with all the $X_a$.

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An $r$-jet $z$ is $V$- (respectively $SV$-, respectively $R$-) sufficient in $C^s$ if all $C^s$ representatives are equivalent in the appropriate sense.

The most fundamental results are the analytic criteria for sufficiency (e.g. that $r$-jets of functions are $R$-sufficient in $C^r$ iff $\|\nabla f\| \geq C\|x\|^{r-1}$ near 0, for some positive constant C), due to Kuiper, Kuo, Lojasiewicz, Bochnak and Kucharz, and Pelczar.

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Finally there are results showing that if sufficiency fails there are in certain cases infinitely many distinct representatives (Bochnak-Kuo, Brodersen and Lefebvre-Pourprix). But an example of Koike-Kucharz, related to the example in Figure 2, showed this is not always the case.

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Suppose that $\mathcal{S}$ is either:
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$z$ is $(t^r)$ for $\mathcal{S}$ iff $z$ is $\mathcal{S}$-sufficient iff $z$ is weakly $\mathcal{S}$-sufficient (resp. $z$ has finitely many $\mathcal{S}$ equivalence classes).

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Results on sufficiency of the jet of a mapping $f : \mathbb{R}^n \to \mathbb{R}^p$ follow by applying parts (2) and (3) of this theorem to the graph of $f$.

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The following theorem is a sample of what can be proved:

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Then the family $X \cap P(t)$ is Whitney equisingular.

# Sun's work on $t^r$ for weighted spaces (1996)

Paunescu proved weighted versions of the $R$ and $V$ sufficiency results in 1993, 1994.

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Sun proved weighted versions of the $t^r$ results of TW in 1999, with most of Paunescu's results as Corollaries.

# Valette's work on $t^r$ bi-Lipschitz and differentiable equisingularity(2009)

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In addition, he proves that the number of Lipschitz types of intersection of smooth direct transversals at a given point is finite when the stratification satisfies the Whitney ($a$) condition.