Transversality and its relation to regular stratifications

Trotman 1978/78 (Invent. Math.)

Stability of transversality to a stratification implies Whitney (a)-regularity!

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A stratification Σ of a closed set $A \subset M$ (C^∞-manifold) is writing $\Sigma = \bigcup_\alpha S_\alpha$, S_α - submanifolds of M, S_α's are disjoint and connected. (strata).

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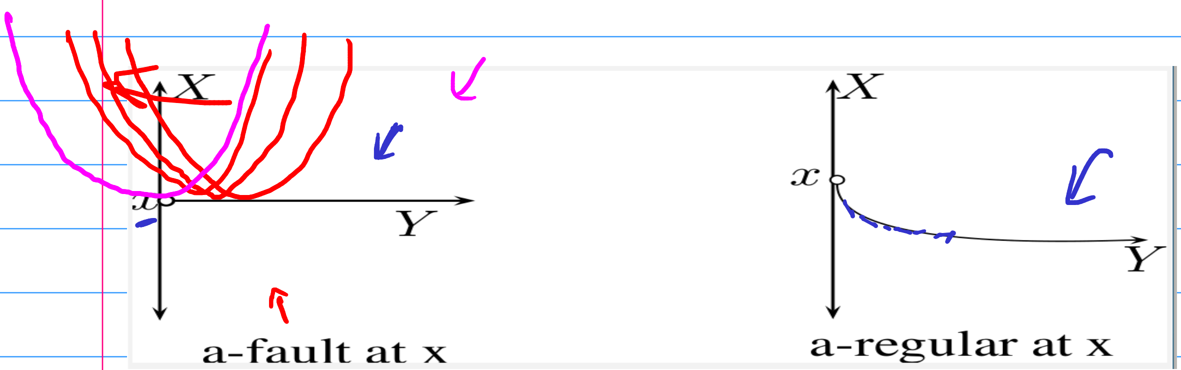
Given Σ,

A stratum Y is a-regular over X at $x \in X \cap \overline{Y}$ if $\forall$ seq. $\{y_n\} \subset Y$ converging to $x \in X$

$$\lim_{n\to\infty} T_{y_n} Y \supset T_x X$$

(if it exists)

If Y is not a-regular over X at $x \in X$, then x is an a-fault.



$y \geq +$ve y axis

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$C^\infty(M,N)$ - Set of all smooth maps from M to N with Whitney strong topology.

A typical nbhd of a map $f \in C^{\infty}(M,N)$ is given by

$$\{ g \in C^{\infty}(M,N) \mid \| f(x) - g(x) \| < \varepsilon(x),\ \varepsilon(x) \text{ is a cont. function on } M \}$$

Denote $f \pitchfork^{C^N} \Sigma$ to say f is transverse to every stratum of Σ.

<u>Stability of Transversality.</u>

<u>Feldman 1965</u> — If $\Sigma \subset N$ is Whitney a-regular, then $\{ f \in C^{\infty}(M,N) \mid f \pitchfork \Sigma \}$ is an open set of $C^{\infty}(M,N)$.

<u>Trotman</u>: (1978/79):- If $\{ f \in C^{\infty}(M,N) / f \pitchfork \Sigma \}$ is open in $C^{\infty}(M,N)$ then Σ is a-regular. $\{$ Given $\dim M \geq \operatorname{cod} \Sigma$ in $N \}$.

Pf: By contradiction,

Need to prove the existence of a seq. $f_n : M \to N$ non transverse to Σ whose lim is transverse, assuming Σ is not a-regular.

(Not a constructive proof)

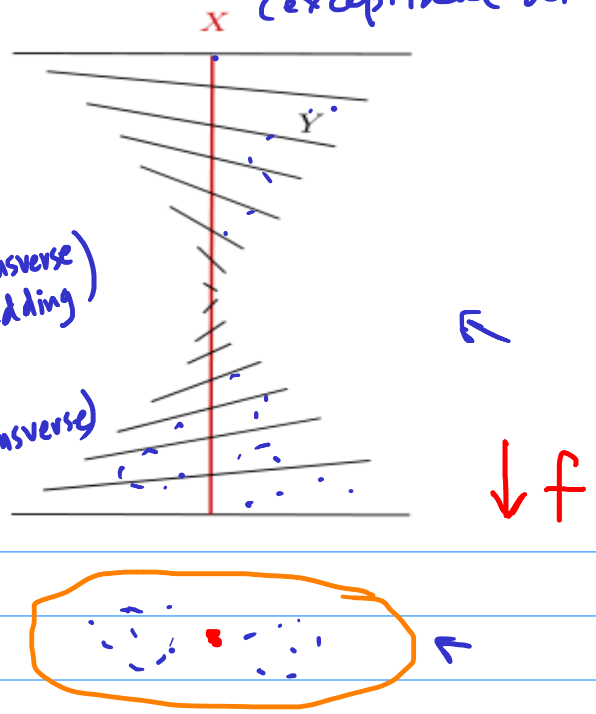
Uses Baire property of $C^{\infty}(M,N)$. and application of Thom <u>Transversality</u> theorem (Denseness).

Thom a_f-regularity :-

Given stra. $\Sigma \subset N$, $f : N \to P$ s.t

rank of f is constant on the strata of Σ, f induces a regular foliation on each strata of Σ. (call $\mathcal{F}_\Sigma$ the foliated stratification)

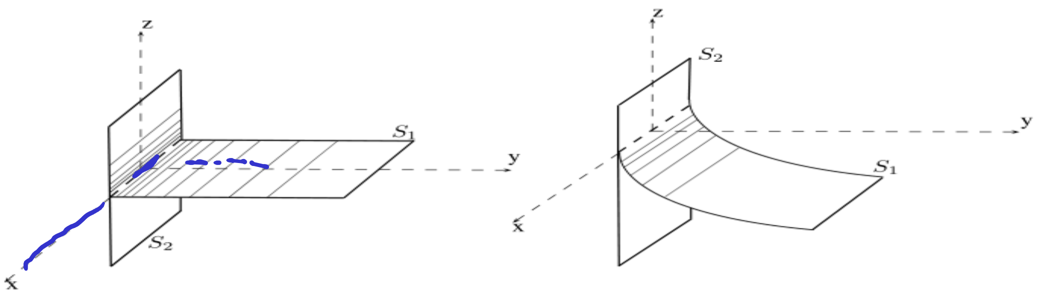
(exceptional divisor)



$M^2 \longrightarrow$ (no transverse embedding)

$M^3 \longrightarrow$ (any map is transverse)

(non a_f-reg. stratification)



(a_f)-regular but not (a)-regular

(a)-regular but not (a_f)-regular

$N = \mathbb{R}^3$

$S^1 = \{z = 0, y > 0\}$

$S^2 = \{y = 0\}$ $\quad f : \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = y + z$

$N = \mathbb{R}^3$

$S^1 = \{y > 0, z < 0, y = z^2\}$

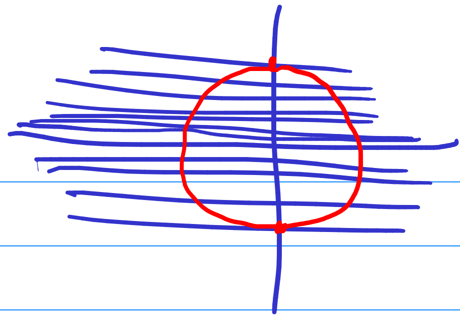
$S_2 = \{y = 0\}$

$f(x,y,z) = y$

Theorem : If the set $\{g \in C^\infty(M,N) \mid g \pitchfork \mathcal{F}_\Sigma\}$ is open in $C^\infty(M,N)$, then Σ is a_f-regular.

Ex: $f: S^1 \to \mathbb{R}^2$

(No map is transverse to foliation)



Thom transverlality does not hold for transversality to foliations.

Thm: Stability of transversality to foliated stratification implies a_f-regularity.

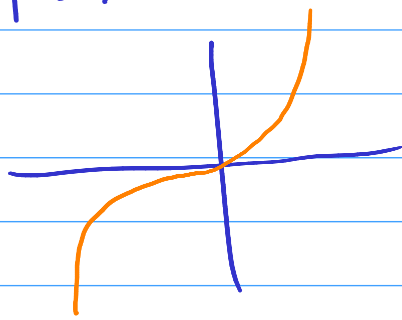
(Still open in the complex case)

Key point

Existence of map $f: M^m \to M^m$ s.t.

$$\left\{ \begin{array}{l} \operatorname{rank} f|_x < m \\ \operatorname{rank} f|_{y \neq x} = m, \end{array} \right\}$$

Ex: $f(x) = x^3$



Complex manifolds and holomorphic map:-

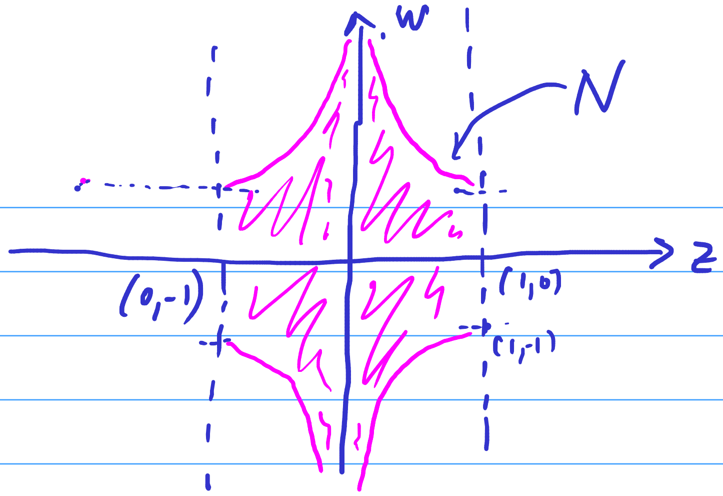
Kaliman and Zaidenberg (1996) (Local version of Thom transversality theorem)

Thom Transversality fails for global holomorphic maps.

Ex: $M = \mathbb{C}$, $N \subset \mathbb{C} \times \mathbb{C}$ given by

$$N = \{(z,w) \mid |z| < 1, |zw| < 1\}.$$

Claim: Any non constant holomorphic map $f: M \to N$ has its image inside w-axis.



Suppose $f(x) = (z(x), w(x))$, z, w are holomorphic

$|z(w)| < 1 \Rightarrow$ By Liouville's thm, z is const., $z(x) = c$, $|c| < 1$.

Suppose $c \neq 0$, then, $|z(x)\, w(x)| < 1$

$\Rightarrow |w(x)| < \frac{1}{|c|}$. $\Rightarrow$ w is const. (contradiction)

$\Rightarrow$ f has its image inside w-axis

Forstneric (2006) - If M is a Stein manifold and N is an Oka manifold. Then, Thom transversality theorem holds.

Converse of Trotman's result holds in the complex case, assuming M is Stein and M is Oka. (Trivedi, 2013)

O-minimal structure $\Leftarrow$ TTT for C^k-def. maps between definable C^k manifolds proved by [Taleloi - 2008].

Problems

① Le Gal and Rolin - C^∞-decomposition does not hold.

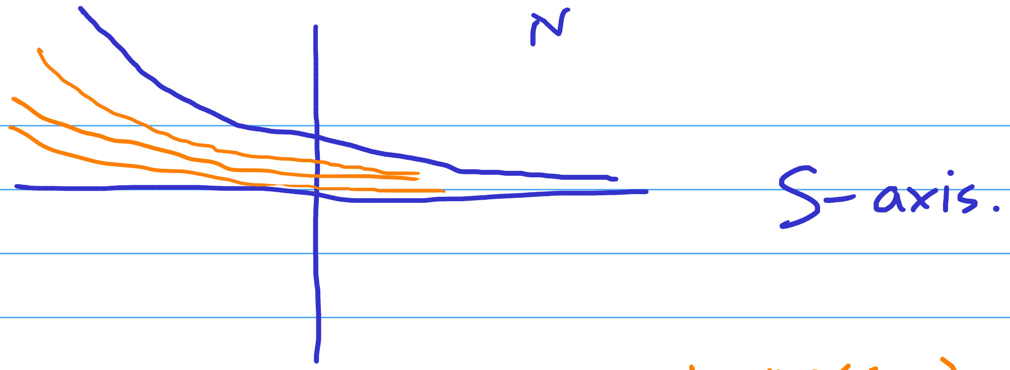
② What topo. to use?

TTT does not hold in semialgebraic setting in the top. induced from Whitney strong topology. Example:

$M = \mathbb{R}$, $N = \mathbb{R}^2$

$f(x) = (x, 0)$

$\varepsilon(x) = e^{-x}$



$\mathcal{N}(f, \varepsilon)$ = no semialgebraic maps in $\mathcal{N}(f, \varepsilon)$.

Assuming that ε is a definable cont. function, Fischer proved.

(Fischer 2008). $\mathcal{N}(f, \varepsilon)$ has enough a lot of definable smooth maps in exponential o-minimal structures.

Nguyen, T. (2020)

An analogue of TTT and Trotman's result hold for o-minimal structures.

Provided:

1. structure is non-polynomially bounded,
2. Exponential
3. Equipped with definable topology.